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'Air' polynomials, Lie point symmetries and a hyperbolic equation

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Abstract

The solution of the problem of one-dimensional, unsteady, isentropic gas flow with the use of Riemann's invariants gives rise to a linear hyperbolic partial differential equation with the ratio of specific heats as an essential parameter. The partial differential equation has $3 + 1 + \infty$ Lie point symmetries. The classical solutions are recovered with the use of the nongeneric symmetries to construct similarity solutions. Further solutions, both polynomial and other, are constructed using the invariants of the Lie point symmetries as seed solutions and the property of mapping solutions into solutions. These solutions are analogous to the well-known heat polynomials.

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1. Introduction

Heat polynomials were introduced in the context of solutions to the heat equation in the last quarter of the nineteenth century [3, 4] and are closely associated with the Hermite polynomials that have found application in the solution of the Schrödinger equation, which is closely connected to the heat equation in a mathematical if not a physical sense. In the century and a quarter since Appell's initial paper, there have been extensions in a number of directions. Some examples are the generalized heat equation in which the second derivative with respect to the space variable is replaced by a higher order derivative [16, 43], multi-dimensional versions of the polynomials [6], a generalization of the radial version of the heat equation in which there is no angular dependence, and the integral dimension of the space is replaced by a non-negative real number [12] and parallel treatments for the linear wave equation [27, 44]. Apart from the final citation, the approach is classical. In the paper of Yehorchenko *et al* [44], conditional symmetry operators are employed. However, the methodology employed there is different to that used in this paper. Maciąg [27] is concerned with the use of wave polynomials in the solution of three-dimensional homogeneous and inhomogeneous wave equations. Here, we are concerned with the construction of polynomial and other solutions.

In this paper, we wish to present some families of polynomials and other functions which arise as closed-form solutions of a hyperbolic equation, which arises in the analysis of a classical problem in unsteady one-dimensional, isentropic gas flow. In this respect, we extend the concept of heat polynomials to another area, hence, the expression 'air' polynomials.

Firstly, we remind the reader of the derivation of the governing equation in terms of Riemann's invariants as the independent variables. We apply the Lie analysis for the possession by this equation of point symmetries. We use the symmetries to generate basic solutions and then the property of the mapping of solutions into solutions to generate further solutions thereby establishing whole families of solutions.

The symmetries which we can actively use for the generation of basic solutions are three in number and possess the Lie algebra $sl(2, R)$. This algebra is ubiquitous in the study of several classes of differential equation. In the first instance, it is characteristic of scalar nth-order ordinary differential equations of maximal symmetry and *a fortiori* of systems of nth-order ordinary differential equations of maximal symmetry. Secondly, still in the realm of ordinary differential equations, the algebra $sl(2, R)$ is the essential identifier of Ermakov systems [8, 19, 37], which have found considerable application in practical matters [18, 24–26] as well as providing an excellent springboard for studies of classes of integrable systems [10, 38–41]. In the more immediately related field of partial differential equations, this algebra is found in the quantal counterparts of the classical Hamiltonian systems associated with linear second-order differential equations and Ermakov–Pinney equations [2, 23], and by simple extension through point transformations to a dazzling variety of evolution equations describing models arising in such divers areas as the conduction of heat, the development of tumours in the brain and the pricing of stocks and other financial instruments in the markets. Some of the studies relating to the algebraic aspects of these several fields may be found in [5, 9, 20, 21, 29, 33].

In the following sections, we see that this classical equation based upon a very classical treatment provides a basis for an analysis of the differential equation concerned in terms of the protocols of modern group analysis. The analysis itself is founded upon the work of Lie, which itself is now already over 130 years old, on continuous symmetry groups and infinitesimal transformations, but the realization of the practical application of the ideas of Lie has really only occurred in the last half century following the pioneering initiatives of Ovsiannikov [34–36]. In this work, we provide a modest contribution to the group theoretical analysis of gas dynamics through the connection of the ideas of invariance under infinitesimal transformations and the existence of polynomial solutions to the heat equation—not to mention related partial differential equations—which go back to the days of Laguerre and Appell [3, 14]. We demonstrate the relationship between these polynomial solutions and the group theoretical properties of the equation we consider. The attraction of this equation is that it does not belong to the class of evolution equations, since it is hyperbolic, generally connected with the ideas of heat polynomials[1].

In the case of one-dimensional, unsteady, isentropic flow of a gas in the absence of viscosity, the governing equations are

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \tag{1.1}$$

[1] Given that the heat equation, as it is commonly called, has so many applications in areas far removed from the concept of heat, the use of the adjectival noun is doubtless unjustified except on historical grounds. As a major source of potential application is in the area of financial mathematics, perhaps the connection with something which is hot is a little unfortunate.

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) = 0 \tag{1.2}$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) + u\frac{\partial}{\partial x}\left(\frac{p}{\rho^\gamma}\right) = 0, \tag{1.3}$$

representing the continuity of mass, conservation of momentum and adiabatic flow, respectively[2]. In the simplest case of a polytropic gas equation, (1.3) is satisfied by the equation of state

$$p = k\rho^\gamma, \tag{1.4}$$

where k is a constant, and one can write the speed of sound in the gas, c, as

$$c^2 = \frac{\mathrm{d}p}{\mathrm{d}\rho} = \frac{\gamma p}{\rho} = \gamma k \rho^{\gamma-1} \tag{1.5}$$

so that the differential relation

$$\frac{\mathrm{d}\rho}{\rho} = \frac{2}{(\gamma-1)}\frac{\mathrm{d}c}{c} \tag{1.6}$$

follows. We eliminate p and ρ from (1.1) and (1.2) in favour of c by the use of (1.4), (1.5) and (1.6) to obtain

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + \frac{(\gamma-1)}{2}c\frac{\partial u}{\partial x} = 0 \tag{1.7}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{2c}{(\gamma-1)}\frac{\partial c}{\partial x} = 0 \tag{1.8}$$

for the continuity and momentum equations, respectively.

By recombination of (1.7) and (1.8) through addition and subtraction we obtain

$$\left(\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right) r = 0 \tag{1.9}$$

$$\left(\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right) s = 0, \tag{1.10}$$

where

$$r = \frac{1}{2}u + \frac{c}{\gamma-1} \qquad \text{and} \qquad s = -\frac{1}{2}u + \frac{c}{\gamma-1} \tag{1.11}$$

are the Riemann invariants.

The characteristic curves of (1.9) and (1.10) may be written in terms of the differential forms

$$\mathrm{d}x = (u+c)\,\mathrm{d}t \qquad \text{and} \qquad \mathrm{d}x = (u-c)\,\mathrm{d}t. \tag{1.12}$$

We introduce the Riemann invariants, r and s, as new independent variables by rewriting the characteristic curves in (1.12) as a pair of first-order linear equations, *videlicet*

$$\frac{\partial x}{\partial s} = (u+c)\frac{\partial t}{\partial s} \qquad \text{and} \qquad \frac{\partial x}{\partial r} = (u-c)\frac{\partial t}{\partial r}. \tag{1.13}$$

From (1.11) we obtain that

$$u + c = \tfrac{1}{2}[(\gamma+1)r + (\gamma-3)s] \qquad \text{and} \qquad u - c = -\tfrac{1}{2}[(\gamma-3)r + (\gamma+1)s]. \tag{1.14}$$

[2] The symbols have the standard meanings of time (t), displacement (x), velocity (u), density (ρ), pressure (p) and the ratio of specific heats (γ).

We may use the requirement of the equality of the mixed derivatives to eliminate x from (1.13) to obtain a single linear second-order partial differential equation for $t(r, s)$. When we take (1.14) into account, this second-order equation is

$$2\frac{(\gamma-1)}{(\gamma+1)}\frac{\partial^2 t}{\partial r\partial s}+\frac{1}{(r+s)}\left(\frac{\partial t}{\partial r}+\frac{\partial t}{\partial s}\right)=0. \tag{1.15}$$

In the case of polytropic gases, it is known that an explicit solution of the initial value problem for (1.15) is possible in terms of the hypergeometric function. In this paper, we consider the solutions obtained by means of the Lie point symmetries of this hyperbolic equation. The classical solutions are recovered using the techniques of Lie's symmetry analysis. We find that there exist classes of polynomial solutions reminiscent of the classical heat polynomials of the heat equation. In addition, we find solutions in terms of rational functions.

2. The symmetry analysis

We rewrite (1.15) in a more standard notation for a hyperbolic equation as

$$(x+y)\frac{\partial^2 u}{\partial x\partial y}+B\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)=0, \tag{2.1}$$

where the parameter B contains the fraction of the ratio of specific heats. Specifically

$$B=\frac{1}{2}\frac{\gamma+1}{\gamma-1}. \tag{2.2}$$

Given the physical origin of the parameter γ, the parameter B is necessarily positive. However, we allow ourselves a certain amount of leeway in the interpretation of the parameter, i.e. B may be any real number.

We note that the parameter B is an essential parameter of (2.1), since it cannot be removed from the equation by rescaling. The different classes of gas—monotomic, diatomic, *etc*—do give rise to different classes of solution, i.e. the mathematical analysis reflects that different categories of gas are *different*. We recall that in the cases that $\gamma = (2N+1)/(2N-1) \Longrightarrow B = N$, where N is a non-negative integer, the solution of (2.1) is known to be possible in terms of elementary functions. Included in these special values are the ratios $5/3$, $7/5$ and $11/9$, which are the values of the ratios of the specific heats for a perfect gas, air and gases produced by some combustion processes.

For B a natural number, the general solution of (2.1) is

$$u=K+\frac{\partial^{N-1}}{\partial x^{N-1}}\frac{f(x)}{(x+y)^N}+\frac{\partial^{N-1}}{\partial y^{N-1}}\frac{g(y)}{(x+y)^N}, \tag{2.3}$$

where f and g are arbitrary functions and is some constant. By a proper choice of the two arbitrary functions and constant, the initial conditions of the problem can be satisfied.

With the aid of program LIE [13, 42] for $B \neq 1$, the Lie point symmetries of (2.1) are easily found to be

$$\Gamma_1=\partial_x-\partial_y \tag{2.4}$$

$$\Gamma_2=x\partial_x+y\partial_y-Bu\partial_u \tag{2.5}$$

$$\Gamma_3=x^2\partial_x-y^2\partial_y-Bu(x-y)\partial_u \tag{2.6}$$

$$\Gamma_4=u\partial_u \tag{2.7}$$

$$\Gamma_5=f(x,y)\partial_u, \tag{2.8}$$

where $f(x, y)$ satisfies the equation

$$(x+y)\frac{\partial^2 f}{\partial x \partial y} + B\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right) = 0, \tag{2.9}$$

which is to be expected since (2.1) is a linear equation[3]. As the case $B = 1$, which corresponds to the value $\gamma = 3$, has some peculiar ramifications, we treat it separately in the next section.

The Lie brackets of the symmetries listed in (2.9) are

$$\begin{array}{lll}
[\Gamma_1, \Gamma_2]_{\mathrm{LB}} = \Gamma_1, & [\Gamma_2, \Gamma_3]_{\mathrm{LB}} = \Gamma_3, & [\Gamma_3, \Gamma_4]_{\mathrm{LB}} = 0, \\
[\Gamma_1, \Gamma_3]_{\mathrm{LB}} = 2\Gamma_2, & [\Gamma_2, \Gamma_4]_{\mathrm{LB}} = 0, & [\Gamma_3, \Gamma_5]_{\mathrm{LB}} = \Gamma_5, \\
[\Gamma_1, \Gamma_4]_{\mathrm{LB}} = 0, & [\Gamma_2, \Gamma_5]_{\mathrm{LB}} = \Gamma_5, & \\
[\Gamma_1, \Gamma_5]_{\mathrm{LB}} = \Gamma_5, & & [\Gamma_4, \Gamma_5]_{\mathrm{LB}} = -\Gamma_5,
\end{array} \tag{2.10}$$

where the brackets with Γ_5 are in a generic sense, i.e. the function $f(x, y)$ in Γ_5 of the right-hand side of the expression for the Lie Bracket need not be the same as for the function in Γ_5 within the bracket on the left-hand side.

The Lie brackets in (2.10) indicate that the algebra of the symmetries is $\{A_1 \oplus A_{3,8}\} \oplus_s \infty A_1$, where we use the Mubarakzyanov classification scheme [28, 30–32]. The algebra $A_{3,8}$ has the common name $sl(2, R)$. The solution symmetries, Γ_5, constitute an infinite-dimensional Abelian subalgebra. In the case of the homogeneity symmetry, Γ_4, the Lie bracket with Γ_5 is essentially the identity. However, the elements of $sl(2, R)$ have the potential to map nontrivially elements of ∞A_1 to other elements of ∞A_1.

The Lie brackets with the 'solution' symmetry, Γ_5, provide a route to the determination of new symmetries in the standard sense that Lie point symmetries map solutions into solutions. In the case of ordinary differential equations this mapping is somewhat trivial since the number of linearly independent solutions is equal to the order of the equation. However, when we are dealing with partial differential equations, the number of solutions is potentially infinite—boundary and initial conditions do make decimation a gentle procedure—and so the role of symmetries in generating solutions is quite critical.

For symmetries to play a critical role in the generation of solutions under the characteristic property of the mapping of solutions into solutions, there is the necessity for a knowledge of symmetries other than our so-called solution symmetries. The procedure for the mapping of a solution into a solution is found in the properties of the Lie bracket as exemplified above. The role of the symmetries in connection with the 'solution' symmetry has been very clearly identified in quantum mechanics (cf [23, 2]) and the same procedure may be applied for evolution equations in general. Whether the solution so generated is useful for the problem as completely defined—i.e. the boundary/initial conditions can be satisfied by the solutions provided by the symmetries—is again another matter. In the absence of other symmetries there does not exist a route for the determination of similarity solutions within the procedure of Lie.

Since (2.1) is a linear partial differential equation, the symmetries Γ_4 and Γ_5 are generic. The symmetries, Γ_1, Γ_2 and Γ_3, are nongeneric and constitute an $sl(2, R)$ subalgebra. We have noted the use of this algebra for the treatment of differential equations, both ordinary and partial, in the introduction.

[3] There are instances in which a nonlinear equation gives rise to the same type of an infinite class of Lie point symmetries based upon the solution of a differential equation which is linear and so obviously not the original equation being analysed. In some cases the linear equation may be obtained by a reasonably obvious point transformation from the original nonlinear equation. However, at least one instance exists for which such a transformation is not only obvious, but the structure of the original equation would appear to make it impossible [21]. For a treatment of a number of nonlinear equations arising in plasma physics, see the recent paper by Cicogna *et al* [7]. Adaptation of the methods used in this paper to such nonlinear equations could be both interesting and useful.

The algebra of the Lie point symmetries of (2.1) is typical of one of the possible classes of algebra associated with the one-dimensional heat equation. In this case the heat equation has a source term of the form $x^{-2}u$ which is typical for the corresponding Ermakov–Pinney ordinary differential equation. As a hyperbolic equation (2.1) can be written as a 1 + 1 wave equation by a simple rotation of the independent variables. As such one would expect to see the Poincaré algebra, or a subset of it, rather than what we have listed in (2.5)–(2.9). The symmetry, Γ_3, does not fit into the scheme of the Poincaré algebra.

3. Special cases

3.1. $B = 1$

When $B = 1$, equation (2.1) has the form

$$(x+y)\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0. \tag{3.1}$$

LIE returns the following results for the Lie point symmetries of (3.1). The general form of the symmetry is

$$\Gamma = a(x)\partial_x + b(y)\partial_y + (uf(x,y) + g(x,y))\partial_u, \tag{3.2}$$

where (x, y) is a solution of (3.1), and the other functions are related according to the two equations

$$(x+y)^2\frac{\partial f}{\partial x} + (x+y)\frac{\mathrm{d}a(a)}{\mathrm{d}x} + a(x) + b(y) = 0 \tag{3.3}$$

$$(x+y)^2\frac{\partial f}{\partial y} + (x+y)\frac{\mathrm{d}b(y)}{\mathrm{d}y} + a(x) + b(y) = 0, \tag{3.4}$$

which is beyond the capability of the program to handle.

We rewrite equations (3.3) and (3.4) as

$$\frac{\partial f}{\partial x} = -\frac{\partial}{\partial x}\left[\frac{a(x)}{x+y} + \frac{b(y)}{x+y}\right] \tag{3.5}$$

$$\frac{\partial f}{\partial y} = -\frac{\partial}{\partial y}\left[\frac{a(x)}{x+y} + \frac{b(y)}{x+y}\right] \tag{3.6}$$

and it is evident that the system is consistent. We integrate (3.5) and substitute the result into (3.6) to obtain

$$f(x,y) = C - \frac{a(x) + b(y)}{x+y}, \tag{3.7}$$

where C is a constant and gives the homogeneity symmetry of (3.1).

In addition to the infinite number of solution symmetries (3.1) possesses a doubly infinite family of symmetries based on the arbitrary functions $a(x)$ and $b(y)$. We may write the three classes of symmetry as

$$\Lambda_1 = a(x)\partial_x + b(y)\partial_y - \frac{a(x)+b(y)}{x+y}u\partial_u \tag{3.8}$$

$$\Lambda_2 = u\partial_u \tag{3.9}$$

$$\Lambda_3 = g(x,y)\partial_u. \tag{3.10}$$

The Lie brackets are

$$\begin{array}{lll} [\Lambda_1, \Lambda_1]_{\mathrm{LB}} = \Lambda_1, & [\Lambda_1, \Lambda_2]_{\mathrm{LB}} = 0, & [\Lambda_1, \Lambda_3]_{\mathrm{LB}} = \Lambda_3, \\ [\Lambda_2, \Lambda_3]_{\mathrm{LB}} = 0, & [\Lambda_3, \Lambda_3]_{\mathrm{LB}} = 0, & \end{array} \tag{3.11}$$

where in the case of the first and last brackets different representatives of the class of symmetries, Λ_1, respectively Λ_3, are taken. In the case of the Lie Brackets between Λ_1 and Λ_3 the result of taking the bracket is generically another element of the class of Λ_3. Naturally it can be zero. The two classes of bracket may be used to generate families of symmetries of the class of Λ_1 and new solutions.

We construct the similarity solution based on the class of Λ_1. The associated Lagrange's system is

$$\frac{\mathrm{d}x}{a(x)} = \frac{\mathrm{d}y}{b(y)} = -\frac{(x+y)\,\mathrm{d}u}{[a(x)+b(y)]u}. \tag{3.12}$$

The invariant associated with the first and second elements of (3.12) is

$$v = \int \frac{\mathrm{d}x}{a(x)} - \int \frac{\mathrm{d}y}{b(y)}. \tag{3.13}$$

To obtain the second invariant we combine the first and second elements of (3.12) so that we have

$$\frac{\mathrm{d}(x+y)}{a+b} = -\frac{(x+y)\,\mathrm{d}u}{(a+b)u} \tag{3.14}$$

and the second invariant is

$$w = (x+y)u. \tag{3.15}$$

When we make the substitution $u = (x+y)^{-1}h(v)$ into (3.1), we obtain the trivial equation $h''(v) = 0$. The constants of integration may be absorbed into the integrals of the arbitrary functions, $a(x)$ and $b(y)$, so that we obtain the solution

$$u(x, y) = \frac{1}{x+y}\left\{\int \frac{\mathrm{d}x}{a(x)} - \int \frac{\mathrm{d}y}{b(y)}\right\}, \tag{3.16}$$

which we identify as the known solution, (2.3), given above up to the additive constant for the case that $N = 1$.

If we take the equation for the case that $N = 1$, *videlicet* (3.1), and differentiate it with respect to x and y in turn, we obtain

$$(x+y)\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial^2 u}{\partial x\partial y}\right) + 2\left(\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x\partial y}\right)\right) = 0. \tag{3.17}$$

This is the equation for the case of $N = 2$ if we take the solution to be $\partial^2 u/\partial x\partial y$. We calculate this derivative for the function

$$u_1(x, y) = \frac{f(x)}{x+y} + \frac{g(y)}{x+y}, \tag{3.18}$$

in which we have replaced the integrals of arbitrary functions with arbitrary functions. Then we obtain

$$u_2(x, y) = -\left\{\frac{\partial}{\partial x}\left[\frac{f(x)}{(x+y)^2}\right] + \frac{\partial}{\partial y}\left[\frac{g(y)}{(x+y)^2}\right]\right\}. \tag{3.19}$$

The general result, (2.3), may be deduced by induction.

In the case that the parameter, B, is a positive integer, the solution of (2.1) is a consequence of the quite different symmetry properties of (2.1) in the case that $B = 1$. (The additive constant is automatic.) We observed above that the integral values of B, at least for small values of the integers, correspond to the physically realized values of the ratio of the specific heats, γ. The ratios mentioned above correspond to $B = 2$, 3 and 5. When $B = 1$, we have $\gamma = 3$. This is not a physical value. It is amusing that the physically relevant solutions can come from a physically irrelevant solution.

3.2. $B = \frac{1}{2}$

We do not dwell upon the value of $B = \frac{1}{2}$ at length since it corresponds to the scarcely physical case of the ratio of specific heats being infinite, but provide an indication of the construction of solutions for this value. There is nothing esoteric about the calculation of further solutions given a seed solution for $B = \frac{1}{2}$. For example, if we take the second seed solution, $\log(x + y)$, for Γ_1 in (4.5) and apply Γ_3 as in (5.1), we obtain the solutions

$$u_1 = (x - y)[1 + B\log(x + y)] \tag{3.20}$$

$$u_2 = (x^2 + y^2)[1 + B\log(x + y)] + B(x - y)^2[2 + B\log(x + y)] \tag{3.21}$$

$$\begin{aligned} u_3 = {} & 2(x^3 - y^3)[1 + B\log(x + y)] + 2B(x - y)(4x^2 - 2xy - y^2) \\ & + B^2(x - y)^2[(x - y) + 2(2x + y)\log(x + y)] \end{aligned} \tag{3.22}$$

from which it is evident that the general form of the solution so generated by Γ_3 has the form

$$u_n = P_n(x, y) + Q_n(x, y)\log(x + y), \tag{3.23}$$

where P and Q are polynomials of homogeneous degree n in x and y. We turn now to the consideration of solutions for general values of the parameter B.

4. The seed solutions

The nontrivial symmetries of (2.1) are (2.4), (2.5) and (2.6). We can use all three to generate solutions invariant under the particular symmetry. The invariants of Γ_1 are found from the solution of the associated Lagrange's system

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{-1} = \frac{\mathrm{d}u}{0} \tag{4.1}$$

and are $v = x + y$ and u. We substitute

$$u(x, y) = g(x + y) = g(v) \tag{4.2}$$

into (2.1) to obtain

$$vg'' + 2Bg' = 0, \tag{4.3}$$

which has the solution

$$g(v) = \begin{cases} M + Nv^{-2B+1}, & B \neq \frac{1}{2}, \\ M + N\log v, & B = \frac{1}{2}, \end{cases} \tag{4.4}$$

where M and N are constants of integration, so that we have the set of similarity solutions

$$u(x, y) = \begin{cases} \{1, (x + y)^{-2B+1}, & B \neq \frac{1}{2}\} \\ \{1, \log(x + y), & B = \frac{1}{2}\} \end{cases}. \tag{4.5}$$

In the case of Γ_3 the associated Lagrange's system is

$$\frac{\mathrm{d}x}{x^2} = \frac{\mathrm{d}y}{-y^2} = \frac{\mathrm{d}u}{-Bu(x - y)}, \tag{4.6}$$

the invariants are

$$v = \frac{1}{x} + \frac{1}{y}, \qquad u = x^{-B} y^{-B} g\left(\frac{1}{x} + \frac{1}{y}\right) \tag{4.7}$$

and

$$u(x, y) = \begin{cases} \left\{(xy)^{-B}, (xy)^{-B}(x+y)^{-2B+1}, \quad B \neq \frac{1}{2}\right\} \\ \left\{(xy)^{-1/2}, (xy)^{-1/2}\log(x+y), \quad B = \frac{1}{2}\right\}. \end{cases} \tag{4.8}$$

We have recorded the solutions for $B = \frac{1}{2}$, but, as they correspond to the value γ being infinite, they are not exactly of physical relevance.

We note that the second-order ordinary differential equation satisfied by the similarity solution, g, is the same for both Γ_1 and Γ_3. The equivalence of these two symmetries in the algebra $sl(2, R)$ is well known.

We have treated Γ_1 and Γ_3 in sequence due to their similarity. We turn now to the odd element of the algebra $sl(2, R)$.

The corresponding expressions for Γ_2 are

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}u}{-Bu}, \tag{4.9}$$

$$v = \frac{y}{x}, \qquad u = x^{-B} g\left(\frac{y}{x}\right), \tag{4.10}$$

where g satisfies the equation

$$v(v+1)g'' + ((2B+1)v+1)g' + B^2 g = 0. \tag{4.11}$$

Equation (4.11) is a hypergeometric equation. The standard form of the hypergeometric equation is ([1, p 562, 11, p 1072])

$$z(1-z)\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + [c - (a+b+1)z]\frac{\mathrm{d}w}{\mathrm{d}z} - abw = 0 \tag{4.12}$$

and (4.11) is converted to the standard form by the simple reflection $z = -v$. However, of more interest is that the parameters a, b and c of (4.12) are specifically related in (4.11) for we have $a = b = B$ and $c = 1$. Rather than invoking the heavy theory of the hypergeometric function, we can make use of the relations between the parameters to provide a treatment by the method of Frobenius (see [17, 396 ff] for an extensive discussion of the method).

We make the ansatz

$$g(v) = \sum_{i=0}^{\infty} a_i v^{i+\sigma} \tag{4.13}$$

and substitute it into (4.11) to obtain

$$\sum_{i=0}^{\infty} (i+\sigma)^2 a_i v^{i+\sigma-1} + \sum_{i=0}^{\infty} (i+\sigma+B)^2 a_i v^{i+\sigma} = 0 \tag{4.14}$$

from which it is evident that the indicial equation is just $\sigma^2 = 0$ or $\sigma = 0(2)$. Consequently it is necessary to make the ansatz

$$g(v) = u_1 \log v + u_2 \tag{4.15}$$

and, after this is substituted into (4.11) and we separate by coefficients of $\log v$ and not-$\log v$, obtain

$$v(v+1)u_1'' + [(2B+1)v+1]u_1' + B^2 u_1 = 0 \tag{4.16}$$

$$v(v+1)u_2'' + [(2B+1)v+1]u_2' + B^2u_2 = -2[(v+1)u_1' + Bu_1], \tag{4.17}$$

where in (4.17) there has been a certain amount of simplification.

We note that formally (4.16) is the same as (4.11) and we make the ansatz (4.13). From (4.14) it is a simple matter to obtain the two-term recurrence relation

$$a_{i+1} = -\left(\frac{i+B}{i+1}\right)^2 a_i, \tag{4.18}$$

which leads to this part of the solution being

$$u_1(v) = \alpha \sum_{n=0}^{\infty} (-1)^n \left(\frac{(B)_n}{n!}\right)^2 v^n, \tag{4.19}$$

where $(B)_n$ is Pochhammer's symbol [1, p 256].

Since the left-hand side of (4.17), the complementary function, is given by (4.19), we now seek a particular solution of (4.17). The recurrence relation is

$$b_{i+1} = -\left(\frac{B+i}{i+1}\right)^2 b_i - \frac{2}{(i+1)^2}\{(i+1)a_{i+1} - (B+i)a_i\}, \tag{4.20}$$

where in the assumed series for u_2 we have replaced a_i with b_i. Since the coefficients in the series of the complementary solution are given by

$$b_{nc} = (-1)^n \left(\frac{(B)_n}{n!}\right)^2 b_0, \tag{4.21}$$

we may write the particular solution as

$$b_{np} = (-1)^n \left(\frac{(B)_n}{n!}\right)^2 b(n) \tag{4.22}$$

so that (4.20) becomes, after a modicum of simplification,

$$b(n+1) = b(n) + 2\left\{\frac{1}{n+1} - \frac{1}{B+n}\right\} \tag{4.23}$$

from which it follows that

$$b(n) = b(0) + 2\sum_{j=0}^{n-1}\left(\frac{1}{j+1} - \frac{1}{B+j}\right) a_0. \tag{4.24}$$

The solution of (4.11) is then

$$\begin{aligned} g(v) = \alpha &\left\{\sum_{n=0}^{\infty} (-1)^n \left(\frac{(B)_n}{n!}\right)^2 v^n \left[\log v + 2\sum_{j=0}^{n-1}\left(\frac{1}{j+1} - \frac{1}{B+j}\right)\right]\right\} \\ &+ \beta\left\{\sum_{n=0}^{\infty} (-1)^n \left(\frac{(B)_n}{n!}\right)^2 v^n\right\}. \end{aligned} \tag{4.25}$$

This solution coincides with that given by Abramowitz and Stegun [1] (15.5.17, p 564) when suitably adjusted. The value $B = \frac{1}{2}$ does not impose a different form of solution. Since the logarithmic term has already entered, perhaps one should not be surprised.

Both solutions in (4.25) can terminate if B is a negative integer. If $B = -N$, which corresponds to the nonphysical $\gamma = (2N-1)/(2N+1) < 1$, the solution has the form

$$g(v) = a_0 P_{N-1}(v)\log v + b_0 Q_{N-1}(v), \tag{4.26}$$

which in the physical case is valid for all nonzero v, since the variables x and y are necessarily positive. More generally the series in (4.25) converge if the dual conditions

$$|v| < \left(\frac{n+1}{n+B}\right)^2 \qquad |v| < \frac{n+1}{n+B} \tag{4.27}$$

are satisfied in the limit as $n \longrightarrow \infty$. The radius of convergence is 1, which is just the case for the expansion of the standard hypergeometric function about the origin[4].

We have devoted more time and space to the solutions corresponding to Γ_2 for two reasons. Firstly they are not as simple as for Γ_1 and Γ_3. This is not an original observation for the additional effort to find the solution corresponding to Γ_2 has been experienced elsewhere [22]. Secondly unlike the solutions for Γ_1 and Γ_3 the solutions obtained are not of an elementary form except for the particular case in which B is a negative integer. In this context it is of interest to note that both solutions terminate for the same value of B. This is not such a common experience. Usually, if one series solution terminates, the other must labour to infinity. The solutions for the autonomous Schrödinger equation for the simple harmonic oscillator are a classic case in point. The contrast is that here the two-term recurrence relation moved one at a time whereas in that case odd and even solutions exist and series in odd and even powers dependent upon an (integral) parameter are highly unlikely to terminate together.

5. Generation of solutions

We commence with the ‘trivial’ solution $f_0 = 1$ of the solution set for Γ_1. The Lie Brackets of Γ_3 and Γ_2 with Γ_5 are

$$[\Gamma_3, \Gamma_5]_{\mathrm{LB}} = \left[x^2\frac{\partial f}{\partial x} - y^2\frac{\partial f}{\partial y} + B(x-y)f\right]\partial_u \tag{5.1}$$

and

$$[\Gamma_2, \Gamma_5]_{\mathrm{LB}} = \left[x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + Bf\right]\partial_u, \tag{5.2}$$

respectively.

In the case of Γ_2 we note that without $-Bu\partial_u$ the effect of the symmetry is just $xf_x + yf_y$. This may be a more attractive form for the generation of solutions in general. However, in the case of the seed solution $f_0 = 1$ to be used here, this form of the similarity symmetry acts as a specific annihilation operator. The form of Γ_2 which we have chosen was dictated by the need for the closure of the algebra of the three nontrivial symmetries. The generic symmetry, Γ_4, has the Lie bracket, $[\Gamma_4, \Gamma_5]_{\mathrm{LB}} = -f\partial_u = -\Gamma_5$, i.e. it is essentially the identity and not of any apparent attraction.

The first few solutions generated from the trivial solution by Γ_3 are

$$u_1 = B(x-y) \tag{5.3}$$

$$u_2 = B(x^2+y^2) + B^2(x-y)^2 \tag{5.4}$$

$$u_3 = 2B(x^3-y^3) + 3B^2(x-y)(x^2+y^2) + B^3(x-y)^3 \tag{5.5}$$

$$\begin{aligned} u_4 = 6B(x^4-y^4) &+ B^2\{3(x^2-y^2)(x^2+y^2) + 2(x-y)(x^3+y^3)\} \\ &+ 6B^3(x-y)^2(x^2+y^2) + B^4(x-y)^4. \end{aligned} \tag{5.6}$$

[4] One notes that a similar reduction can be made if one chooses $v = x/y$ rather than y/x. Consequently the effect of the restriction, $|y/x| < 1$, is not as great as the ratio test implies.

We observe that the solutions generated are homogeneous of degree n in x and y for u_n. Consequently the Lie Bracket of Γ_2 and Γ_5 has the nature of an eigenvalue relationship in that

$$[\Gamma_2, \Gamma_5]_{\text{LB}} = (B+n)\Gamma_5. \tag{5.7}$$

There are variations on the taking of this Lie bracket which one may use. If we take $\bar{\Gamma}_2 = x\partial_x + y\partial_y$ and act on f_n, we have

$$\bar{\Gamma}_2 f_n = n f_n, \tag{5.8}$$

i.e. $\bar{\Gamma}_2$ plays a role similar to that of ∂_t in the algebraic treatment of the time-dependent Schrödinger equation [2, 23] and is truly an eigenvalue equation. On the other hand the action of Γ_2 provides a different result in that

$$\Gamma_2 f_n = -(B-n) f_n \tag{5.9}$$

and acts almost as a convolutive eigenvalue operator. In the case that B is a positive integer there is eventually a zero eigenvalue for what could be a very nontrivial eigenfunction. In this case the solution generated by Γ_3 from the trivial solution compatible with Γ_1 is also a solution due to Γ_2 as was discussed in the previous section. Obviously it is not the general solution associated with Γ_2 since there is the further requirement of compatibility with the annihilation property of the repeated action of Γ_1.

A less trivial starting point is the first solution due to Γ_3, *videlicet*

$$u_0 = (xy)^{-B}. \tag{5.10}$$

The Lie Bracket of Γ_1 with Γ_5 with u_0 as f leads to a sequence of solutions of which we give the first few. Note that, since

$$[\Gamma_1, \Gamma_5]_{\text{LB}} = \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)\partial_u, \tag{5.11}$$

the calculation is the same as taking $\Gamma_1 f$. We obtain

$$\begin{aligned}
u_1 &= B(x-y)(xy)^{-B-1} \\
u_2 &= B(B+1)(x-y)^2(xy)^{-B-2} + 2B(xy)^{-B-1} \\
u_3 &= B(B+1)(B+2)(x-y)^3(xy)^{-B-3} + 6B(B+1)(x-y))(xy)^{-B-2} \\
u_4 &= B(B+1)(B+2)(B+3)(x-y)^4(xy)^{-B-4} + 12B(B+1)(B+2) \\
&\quad \times (x-y)^2(xy)^{-B-3} + 12B(B+1)(xy)^{-B-2}
\end{aligned} \tag{5.12}$$

and in general one can write

$$u_n = (\Gamma_1)^n u_0. \tag{5.13}$$

The general second solution for Γ_3 is

$$u_0 = (xy)^{-B}(x+y)^{-2B+1}. \tag{5.14}$$

We use Γ_1 to generate the first few of the sequence of solutions, *videlicet*

$$\begin{aligned}
u_1 &= B(x-y)(xy)^{-B-1}(x+y)^{-2B+1} \\
u_2 &= (2Bxy + B(B+1)(x-y)^2)(xy)^{-B-2}(x+y)^{-2B+1} \\
u_3 &= B(B+1)(6(x-y)xy - (B+2)(x-y)^2)(xy)^{-B-3}(x+y)^{-2B+1},
\end{aligned}$$

which, apart from the factor $(x+y)^{-2B+1}$, is the same sequence as that which is generated by the action of Γ_1 on the seed solution $(xy)^{-B}$ since $\Gamma_1(x+y) = 0$.

It should be clear that a variety of solutions can be presented by means of the use of the symmetries to act as ladder operators on the seed solutions.

6. A general homogeneous polynomial solution

The solutions generated by Γ_3 are homogeneous polynomials. The formulæ as presented in (5.3)–(5.6) do not suggest an obvious form for the homogeneous polynomial of the nth-degree solution. To find the general form of the polynomial we make the substitution

$$u_n = \sum_{i=0}^{n} a_i x^i y^{n-i} \tag{6.1}$$

into (2.1) and make some rearrangements of the terms to obtain

$$\sum_{i=0}^{n} a_i(B+i)(n-i)x^i y^{n-i-1} + \sum_{i=0}^{n} a_{i+1}(i+1)(B+n-i-1)x^i y^{n-i-1} = 0 \tag{6.2}$$

and the two-term recurrence relation for the coefficients of the polynomial

$$a_{i+1} = -\frac{(B+i)(n-i)}{(i+1)(B+n-i-1)}a_i, \qquad i = 0, 1, \ldots, n-1. \tag{6.3}$$

Now that we have the recurrence relation for the coefficients of the polynomial we may write the general formula as

$$u_n = \sum_{i=0}^{n} (-1)^i \frac{n!}{(n-i)!i!} \frac{(B-1)_i}{(B+n-i)_i} x^i y^{n-i}, \tag{6.4}$$

where again $(q)_n$ indicates Pochhammer's symbol.

The solution, (6.4), has been obtained in the spirit of the solutions generated by the action of Γ_3 on the solution symmetry (Γ_5) containing the trivial solution, $u(x, y) = 1$. What we have done here is to look for a new way to generate the coefficients of the polynomial. That solutions of other structure are possible is obvious from the solutions generated in the previous section.

The second, nontrivial, solution of (4.5) for general values of the parameter B provides another starting point for a family of solutions generated by Γ_3. The first few solutions so generated are

$$u_0 = (x+y)^{-2B+1} \tag{6.5}$$

$$u_1 = (1-B)(x-y)u_0 \tag{6.6}$$

$$u_2 = (1-B)[(x-y)^2 + (x^2+y^2)]u_0. \tag{6.7}$$

These few solutions are suggestive of a structure for a general solution of this genre. This is a homogeneous polynomial times the base solution, u_0. We make the substitution

$$u_n = \left(\sum_{i=0}^{n} a_i x^i y^{n-i}\right)(x+y)^{-2B+1} \tag{6.8}$$

into (2.1) to obtain the *three*-term recurrence relation

$$a_{i+2} = \frac{1}{n-i-1-B}\{(i+1-B)a_i + [2(i+1)-n]a_{i+1}\}, \tag{6.9}$$

which is not amenable to give a solution of as simple a form as obtained in (6.4). We note that Γ_3 acts as an annihilation operator for the similarity solution in the case that $B = 1$.

7. Conclusion

The source problem for the differential equation at the heart of this paper, (2.1), is a classic problem in fluid mechanics. This equation, from which the solution to the original system, (1.1)–(1.3), flows, is a hyperbolic equation. In terms of the Lie symmetry analysis, it differs more than somewhat from the usual expectation of the symmetries of a hyperbolic equation. The $1+1$ wave equation, the closest analogue, has a doubly infinite set of Lie point symmetries whereas (2.1) has an algebraic structure of strongly reminiscent of the $1 + 1$ heat equation with a source term of Ermakov–Pinney form except in the specific instance that the parameter $B = 1$, which corresponds to the nonphysical $\gamma = 3$. Then the algebraic structure differs markedly from both the heat equation and the wave equation.

The symmetries in the case $B = 1$ not only lead to the known solution for this case but also to the solutions known for $B = N$, where N is a positive integer.

More generally we were able to construct both polynomial and other solutions for any value of B. In the case of B a positive integer the other solutions are rational. Otherwise they are algebraic or transcendental depending upon whether B is rational or irrational.

All of the functions generated are solutions of (2.1). In a manner similar to that found with the study of heat polynomials, one could contemplate the construction of polynomials which are not solutions of (2.1), but rather the coefficients of nonpolynomial functions with the products being solutions of (2.1). This is in the same spirit as the solution of the Schrödinger equation for the simple harmonic oscillator. The Hermite polynomials are solutions not of the Schrödinger equation but of a related equation obtained by the removal of an exponential term containing both the time and space variables. We look forward to the investigation of the properties of such polynomials and their potential applications.

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